The Gross-Neveu Model and the Supersymmetric and Non–Supersymmetric Nambu–Jona–Lasinio Model in a Magnetic Field

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The infrared dynamics in the $(3 + 1)$–dimensional supersymmetric and non–supersymmetric Nambu–Jona–Lasinio model in a constant magnetic field is studied. While at strong coupling the dynamics in these two models is essentially different, it is shown that the models become equivalent at weak coupling. In particular, at weak coupling, as the strength of the magnetic field goes to infinity, both the supersymmetric and non–supersymmetric Nambu–Jona–Lasinio models are reduced to a continuum set of independent $(1+1)$–dimensional Gross–Neveu models, labeled by the coordinates in the plane perpendicular to the magnetic field. The relevance of these results for cosmological models based on supersymmetric dynamics is pointed out.

I. INTRODUCTION.

It has been recently shown [1–4] that a constant magnetic field in $2 + 1$ and $3 + 1$ dimensions is a strong catalyst of dynamical chiral symmetry breaking, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions.

The essence of the effect is the dimensional reduction $D \to D - 2$ (i.e. $2 + 1 \to 0 + 1$ and $3 + 1 \to 1 + 1$) in the infrared dynamics of the fermion pairing in a magnetic field. The physical reason of this reduction is the fact that the motion of charged particles is restricted in those directions that are perpendicular to the magnetic field. This is in turn connected with the point that, at weak coupling between fermions, the fermion pairing, leading to the chiral condensate, is mostly provided by fermions from the lowest Landau level (LLL) whose dynamics are $(D - 2)$–dimensional.

In this paper, we shall further clarify the effect of the dynamical reduction, studying in detail the infrared dynamics in the $(3+1)$–dimensional Nambu–Jona–Lasinio (NJL) model in a magnetic field. We shall consider both the ordinary (non–supersymmetric) and supersymmetric versions of the NJL model. In particular, we shall show that in the "continuum" limit, when both the strength of the magnetic field and the ultraviolet cutoff go to infinity, both the non–supersymmetric and supersymmetric (weakly coupling) NJL models with N_c colors are reduced to a continuum set of independent $(1+1)$–dimensional Gross-Neveu (GN) models [5], labeled by coordinates $\mathbf{x}_\perp$ in the plane perpendicular to the magnetic field $\mathbf{B}$. The number of colors in the GN models is $\tilde{N}_c = (\pi/2C)N_c$, where C is $C = \Lambda^2/|eB|$ in the "continuum" limit (here Λ is the ultraviolet cutoff). As will be shown in Sec.3, the factor $\pi/2C$ is proportional to a (local) magnetic flux attached to each point in the $x_\perp$-plane.

On the other hand, at strong coupling, the dynamics in the supersymmetric and non–supersymmtric NJL models are very different.

Recall that there is no spontaneous chiral symmetry breaking in the supersymmetric NJL model [6]. An external magnetic field changes the situation dramatically: chiral symmetry breaking occurs for any value of the coupling constant in this model. This agrees with the general conclusion of Refs. [2–4] that the effect of the catalysis of chiral symmetry breaking by a magnetic field in $3 + 1$ dimensions is a universal, model–independent effect.

As was already shown in Ref. [4], the dimensional reduction $3 + 1 \to 1 + 1$ in the dynamics of the fermion pairing in a (finite) magnetic field is consistent with spontaneous symmetry breaking. Recall that, due to the Mermin–Wagner–Coleman (MWC) theorem [7], there cannot be spontaneous breakdown of continuous symmetries at $D = 1 + 1$. The MWC theorem is based on the fact that gapless Nambu–Goldstone (NG) bosons cannot exist in $1 + 1$ dimensions. However, in a magnetic field, the reduction $3 + 1 \to 1 + 1$ takes place (in the infrared region) only for propagators of charged particles: it reflects the fact that the motion of charged particles is restricted in the directions perpendicular to the magnetic field. On the other hand, NG bosons connected with spontaneous chiral symmetry breaking are

neutral and therefore their propagators have $(3 + 1)$–dimensional form [4][1]. This in turn implies that the effect of spontaneous chiral symmetry breaking does not contradict the MWC theorem.

The interplay between the GN model and the NJL model in a magnetic field established in this paper further clarifies this issue. As was mentioned above, the NJL model is reduced to a set of the GN models only as the strength of the magnetic field goes to infinity. As we shall show in Sec.4, at finite $|eB|$, the dynamics of the NJL model in a magnetic field are in a sense similar to the dynamics in the $(2 + \epsilon)$–dimensional GN model: the magnetic length $l = |eB|^{-1/2}$ plays here the role of the (physical) ϵ–parameter which is an infrared regulator.

As was already pointed out in Refs. [1–4], there may be interesting applications of this effect in cosmology as well as in particle and condensed matter physics. The results of the present paper may be particularly relevant for cosmological scenarios based on supersymmetric dynamics [9].

The paper is organized as follows. In Section 2 we, for completeness, derive the effective action in the GN model. In Sections 3 and 4 we establish the connection between the GN model and the NJL model in a magnetic field as $|eB| \to \infty$. In Section 5 we consider the dynamics of the supersymmetric NJL model in a magnetic field. In Section 6 we summarize the main results of the paper and discuss possible applications of these results and as well as the possibility of their extension to inhomogeneous magnetic field configurations. In the Appendix some useful formulas and relations are derived.

II. EFFECTIVE ACTION IN THE GROSS–NEVEU MODEL.

In this section, for completeness, we shall derive the effective action for the GN model. The Lagrangian density of the GN model is:

$$\mathcal{L}_{GN} = \frac{1}{2} \left[\bar{\Psi}, (i\gamma^\mu \partial_\mu) \Psi \right] + \frac{\tilde{G}}{2} \left[(\bar{\Psi}\Psi)^2 + (\bar{\Psi} i\gamma^5 \Psi)^2 \right] \tag{1}$$

where $\mu = 0, 1$ and the fermion field carries an additional "color" index $\tilde{\alpha} = 1, 2, \ldots, \tilde{N}_c$ (for simplicity, we consider the case of the chiral $U_L(1) \times U_R(1)$ symmetry). The theory is equivalent to the theory with the Lagrangian density

$$\mathcal{L}'_{GN} = \frac{1}{2} \left[\bar{\Psi}, (i\gamma^\mu \partial_\mu) \Psi \right] - \bar{\Psi}(\sigma + i\gamma^5 \pi)\Psi - \frac{1}{2\tilde{G}} \left(\sigma^2 + \pi^2 \right). \tag{2}$$

The Euler–Lagrange equations for the auxiliary fields σ and π take the form of constraints:

$$\sigma = -\tilde{G}\bar{\Psi}\Psi, \qquad \pi = -\tilde{G}\bar{\Psi} i\gamma^5 \Psi, \tag{3}$$

and the Lagrangian density (2) reproduces Eq.(1) upon application of the constraints (3). The effective action for the composite fields σ and π can be obtained by integrating over fermions in the path integral. It is given by the standard relation:

$$\Gamma_{GN}(\sigma, \pi) = \tilde{\Gamma}_{GN}(\sigma, \pi) - \frac{1}{2\tilde{G}} \int d^2 x (\sigma^2 + \pi^2), \tag{4}$$

$$\tilde{\Gamma}_{GN}(\sigma, \pi) = -iTrLn \left[i\gamma^\mu \partial_\mu - (\sigma + i\gamma^5 \pi) \right]. \tag{5}$$

The low energy quantum dynamics are described by the path integral (with the integrand $\exp(i\Gamma_{GN})$) over the composite fields σ and π. As $\tilde{N}_c \to \infty$, the path integral is dominated by the stationary points of the action: $\delta\Gamma_{GN}/\delta\sigma = \delta\Gamma_{GN}/\delta\pi = 0$. We will analyze the dynamics by using the expansion of the action Γ_{GN} in powers of derivatives of the composite fields.

We begin the calculation of Γ_{GN} by calculating the effective potential V_{GN}. Since V_{GN} depends only on the $U_L(1) \times U_R(1)$–invariant $\rho^2 = \sigma^2 + \pi^2$, it is sufficient to consider a configuration with $\pi = 0$ and σ independent of x. Then we find from Eqs. (4) and (5):

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[1] The Lorentz invariance is broken by a magnetic field in this problem. By the $(3 + 1)$–dimensional form, we understand that the denominator of the propagators depends on energy and all the components of the center–of–mass momentum, i.e. $\mathcal{D}(P) \sim (P_0^2 - C_\parallel P_\parallel^2 - C_\perp \mathbf{P}_\perp^2)^{-1}$ with $C_\parallel, C_\perp \neq 0$.

$$V_{GN}(\rho) = \frac{\rho^2}{2\tilde{G}} - \tilde{N}_c \int \frac{d^2k}{(2\pi)^2} \ln\left(\frac{k^2 + \rho^2}{k^2}\right) =$$

$$= \frac{\rho^2}{2\tilde{G}} - \frac{\tilde{N}_c \rho^2}{4\pi}\left[\ln\frac{\Lambda^2}{\rho^2} + 1\right], \tag{6}$$

where the integration is done in Euclidean region (Λ is an ultraviolet cutoff). As is known, in the GN model, the equation of motion $dV_{GN}/d\rho = 0$ has a nontrivial solution $\rho = \bar{\sigma} \equiv m_{dyn}$ for any value of the coupling constant $\tilde{G}$. Then the potential V_{GN} can be rewritten as

$$V_{GN}(\rho) = \frac{\tilde{N}_c \rho^2}{4\pi}\left[\ln\frac{\rho^2}{m_{dyn}^2} - 1\right], \tag{7}$$

where

$$m_{dyn}^2 = \Lambda^2 \exp\left(-\frac{2\pi}{\tilde{N}_c \tilde{G}}\right). \tag{8}$$

Due to the MWC theorem [7], there cannot be spontaneous breakdown of continuous symmetries at $D = 1 + 1$. The parameter m_{dyn} is an order parameter of chiral symmetry breaking only in leading order in $1/\tilde{N}_c$ (this reflects the point that the MWC theorem is not applicable to systems with $\tilde{N}_c \to \infty$ [8]). In the exact GN solution, spontaneous chiral symmetry breaking is washed out by interactions (strong fluctuations) of would–be NG bosons π (i.e. after integration over π and σ in the path integral). The exact solution in this model presumably corresponds to the realization of the Berezinsky–Kosterlitz–Thouless (BKT) phase: though chiral symmetry is not broken in this phase, the parameter m_{dyn} still defines the fermion mass, and the would–be NG boson π transforms into a BKT gapless excitation [8].

Let us now turn to calculating the kinetic term in Γ_{GN}. The chiral $U_L(1) \times U_R(1)$ symmetry implies that the general form of the kinetic term is

$$\mathcal{L}_{GN}^{(k)} = \frac{f_1^{\mu\nu}}{2}(\partial_\mu \rho_j \partial_\nu \rho_j) + \frac{f_2^{\mu\nu}}{\rho^2}(\rho_j \partial_\mu \rho_j)(\rho_i \partial_\nu \rho_i) \tag{9}$$

where $\boldsymbol{\rho} = (\sigma, \pi)$ and $f_1^{\mu\nu}$, $f_2^{\mu\nu}$ are functions of ρ^2. To find the functions $f_1^{\mu\nu}$ and $f_2^{\mu\nu}$, one can use different methods. We utilize the same method as in Ref. [4] (see Appendix A in that paper). The result is:

$$f_1^{\mu\nu}(\rho^2) = -\frac{i}{2} \int \frac{d^2k}{(2\pi)^2} tr\left[S(k)i\gamma_5 \frac{\partial^2 S(k)}{\partial k_\mu \partial k_\nu} i\gamma_5\right], \tag{10}$$

$$f_2^{\mu\nu}(\rho^2) = -\frac{i}{4} \int \frac{d^2k}{(2\pi)^2} tr\left[S(k)\frac{\partial^2 S(k)}{\partial k_\mu \partial k_\nu} - S(k)i\gamma_5 \frac{\partial^2 S(k)}{\partial k_\mu \partial k_\nu} i\gamma_5\right], \tag{11}$$

with $S(k) = i(k^\mu \gamma_\mu + \rho)/(k^2 - \rho^2)$. The explicit form of these functions is:

$$f_1^{\mu\nu} = g^{\mu\nu}\frac{\tilde{N}_c}{4\pi\rho^2}, \qquad f_2^{\mu\nu} = -g^{\mu\nu}\frac{\tilde{N}_c}{12\pi\rho^2} \tag{12}$$

III. THE INTERPLAY BETWEEN THE GN MODEL AND THE NJL MODEL IN A MAGNETIC FIELD

In this section, we compare the effective actions in the GN model and in the NJL model in a magnetic field, and we establish a rather interesting connection between these two models.

The analog of the Lagrangian density (2) in the NJL model in a magnetic field is

$$\mathcal{L}' = \frac{1}{2}\left[\bar{\Psi}, (i\gamma^\mu D_\mu)\Psi\right] - \bar{\Psi}(\sigma + i\gamma^5 \pi)\Psi - \frac{1}{2G}\left(\sigma^2 + \pi^2\right) \tag{13}$$

where $D_\mu = \partial_\mu - ieA_\mu^{ext}$, $A_\mu^{ext} = Bx^2\delta_\mu^3$ (the magnetic field is in $+x_1$ direction).

In leading order in $1/N_c$, the effective action in the NJL model in a magnetic field is derived in Refs. [2,4]. The effective potential and the kinetic term are ($\boldsymbol{\rho} = (\sigma, \pi)$):

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$$V(\rho) = \frac{\rho^2}{2G} + \frac{N_c}{8\pi^2}\left[\frac{\Lambda^4}{2} + \frac{1}{3l^4}\ln(\Lambda l)^2 + \frac{1-\gamma-\ln 2}{3l^4} - (\rho\Lambda)^2 + \frac{\rho^4}{2}\ln(\Lambda l)^2\right.$$

$$\left. + \frac{\rho^4}{2}(1-\gamma-\ln 2) + \frac{\rho^2}{l^2}\ln\frac{(\rho l)^2}{2} - \frac{4}{l^4}\zeta'\left(-1, \frac{(\rho l)^2}{2}+1\right)\right] + O\left(\frac{1}{\Lambda}\right), \tag{14}$$

$$\mathcal{L}^{(k)} = \frac{f_1^{\mu\nu}}{2}(\partial_\mu\rho_j\partial_\nu\rho_j) + \frac{f_2^{\mu\nu}}{\rho^2}(\rho_j\partial_\mu\rho_j)(\rho_i\partial_\nu\rho_i) \tag{15}$$

with $f_1^{\mu\nu}$ and $f_2^{\mu\nu}$ being diagonal tensors:

$$f_1^{00} = -f_1^{11} = \frac{N_c}{8\pi^2}\left[\ln\frac{(\Lambda l)^2}{2} - \psi\left(\frac{(\rho l)^2}{2}+1\right) + \frac{1}{(\rho l)^2} - \gamma + \frac{1}{3}\right],$$

$$f_1^{22} = f_1^{33} = -\frac{N_c}{8\pi^2}\left[\ln\frac{\Lambda^2}{\rho^2} - \gamma + \frac{1}{3}\right],$$

$$f_2^{00} = -f_2^{11} = -\frac{N_c}{24\pi^2}\left[\frac{(\rho l)^2}{2}\zeta\left(2, \frac{(\rho l)^2}{2}+1\right) + \frac{1}{(\rho l)^2}\right], \tag{16}$$

$$f_2^{22} = f_2^{33} = -\frac{N_c}{8\pi^2}\left[(\rho l)^4\psi\left(\frac{(\rho l)^2}{2}+1\right) - 2(\rho l)^2\ln\Gamma\left(\frac{(\rho l)^2}{2}\right)\right.$$

$$\left. -(\rho l)^2\ln\frac{(\rho l)^2}{4\pi} - (\rho l)^4 - (\rho l)^2 + 1\right].$$

Here G is the NJL coupling constant, N_c is the number of colors, $\zeta(\nu, x)$ is the generalized Riemann zeta function, $\zeta'(\nu, x) = \partial\zeta(\nu, x)/\partial\nu$, $\gamma \approx 0.577$ is the Euler constant, $\psi(x) = d\left(\ln\Gamma(x)\right)/dx$, and $l \equiv |eB|^{-1/2}$ is the magnetic length. The gap equation $dV/d\rho = 0$ is[2]

$$\rho\Lambda^2\left(\frac{1}{g}-1\right) = -\rho^3\ln\frac{(\Lambda l)^2}{2} + \gamma\rho^3 + \frac{\rho}{l^2}\ln\frac{(\rho l)^2}{4\pi} + \frac{2\rho}{l^2}\ln\Gamma\left(\frac{(\rho l)^2}{2}\right) + O\left(\frac{1}{\Lambda}\right), \tag{17}$$

where the dimensionless coupling constant $g = N_c G\Lambda^2/4\pi^2$. In the derivation of this equation, we used the relations [10]:

$$\frac{\partial}{\partial x}\zeta(\nu, x) = -\nu\zeta(\nu+1, x), \tag{18}$$

$$\frac{\partial}{\partial\nu}\zeta(\nu, x)\bigg|_{\nu=0} = \ln\Gamma(x) - \frac{1}{2}\ln 2\pi, \quad \zeta(0, x) = \frac{1}{2} - x. \tag{19}$$

As $B \to 0$ ($l \to \infty$), we recover the known gap equation in the NJL model (for a review see Ref. [11]):

$$\rho\Lambda^2\left(\frac{1}{g}-1\right) = -\rho^3\ln\frac{\Lambda^2}{\rho^2}. \tag{20}$$

This equation admits a nontrivial solution only if g is supercritical, $g > g_c = 1$ (as Eq.(13) implies, a solution to the gap equation, $\rho = \bar{\sigma}$, coincides with the fermion dynamical mass, $\bar{\sigma} = m_{dyn}$). As was shown in Refs. [2,4], at $B \neq 0$, a non–trivial solution exists for all $g > 0$.

Let us consider the case of small subcritical g, $g \ll g_c = 1$, in detail. A solution is seen to exist for this case if ρl is small. Specifically, for $g \ll 1$, the left–hand side of Eq.(17) is positive. Since the first term of the right–hand side in this equation is negative, we conclude that a non–trivial solution to this equation may exist only for

$$\rho^2\ln(\Lambda l)^2 \ll \frac{1}{l^2}\ln\frac{1}{(\rho l)^2} \tag{21}$$

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[2]In this paper we consider the case of a large ultraviolet cutoff: $\Lambda^2 \gg \bar{\sigma}^2$, $|eB|$, where $\bar{\sigma}$ is a minimum of the potential V.

$(\Gamma(\rho^2 l^2/2) \approx 2/(\rho l)^2)$. We then find the solution:

$$m_{dyn}^2 \equiv \bar{\sigma}^2 = \frac{|eB|}{\pi} \exp\left(-\frac{4\pi^2(1-g)}{|eB|N_c G}\right) = \frac{|eB|}{\pi} \exp\left(-\frac{(1-g)\Lambda^2}{g|eB|}\right). \tag{22}$$

Actually, since Eq.(22) implies that condition (21) is violated only if $(1-g) \lesssim |eB|/\Lambda^2$, the expression (22) is valid for all g outside that (scaling) region near the critical value $g_c = 1$. Note that in the scaling region $(g \to g_c - 0)$ the expression for m_{dyn}^2 is different [2,4]:

$$m_{dyn}^2 \simeq |eB| \frac{\ln\left[(\ln \Lambda^2 l^2)/\pi\right]}{\ln \Lambda^2 l^2}. \tag{23}$$

Let us compare relation (22) with relation (8) for the dynamical mass in the GN model. The similarity between them is evident: $|eB|$ and $|eB|G$ in Eq.(22) play the role of an ultraviolet cutoff and the dimensionless coupling constant $\tilde{G}$ in Eq.(8). Let us discuss this connection and show that it is intimately connected with the dimensional reduction $3 + 1 \to 1 + 1$ in the dynamics of the fermion pairing in a magnetic field.

Eq.(8) implies that the GN model is asymptotically free, with the bare coupling constant $\tilde{G} = 2\pi/\tilde{N}_c \ln(\Lambda^2/m_{dyn}^2) \to 0$ as $\Lambda \to \infty$. Let us now consider the following limit in the NJL model in a magnetic field: $|eB| \to \infty$, $\Lambda^2/|eB| = C \gg 1$. Then relation (22), which can be rewritten as

$$m_{dyn}^2 = \frac{\Lambda^2}{C\pi} \exp\left(-\frac{C(1-g)}{g}\right), \tag{24}$$

implies that the behavior of the bare coupling constant g must be

$$g \simeq \frac{C}{\ln(\Lambda^2/C\pi m_{dyn}^2)} \to 0, \tag{25}$$

in order to get a finite value for m_{dyn}^2 in this limit. Thus in this "continuum" limit, we recover the same behavior for the coupling g in the NJL model as for the coupling constant $\tilde{G}$ in the GN model.

Let us now compare the effective potentials in these two models. At first glance, the expressions (6) and (14) for the effective potentials in these models look very different: the character of ultraviolet divergences in 1+1 and 3+1 dimensional theories is essentially different. However, using Eqs.(18) and (19), the expression (14) can be rewritten, for small ρl, as

$$V(\rho) = V(0) + \frac{N_c \rho^2}{8\pi^2 l^2}\left[(\Lambda l)^2 \left(\frac{1}{g} - 1\right) - 1 + \ln(\pi \rho^2 l^2) + \right.$$

$$\left. + \frac{(\rho l)^2}{2} \ln \frac{(\Lambda l)^2}{2} + O((\rho l)^4)\right]. \tag{26}$$

Then, expressing the coupling constant g through m_{dyn} from Eq.(22), we find that

$$V(\rho) = V(0) + \frac{N_c |eB|}{8\pi^2} \rho^2 \left[\ln \frac{\rho^2}{m_{dyn}^2} - 1 + O((\rho l)^2)\right]. \tag{27}$$

Here we used the fact that, because of Eq.(21), the ratio $(\rho l)^2$ is small near the minimum $\rho = m_{dyn}$.

The expressions (7) and (27) for the potentials in these two models look now quite similar. There is however an additional factor $|eB|/2\pi$ in the expression (27). Moreover, the field ρ, which depends on the two coordinates x_0 and x_1 in the GN model, depends on the four coordinates x_0, x_1, x_2 and x_3 in the NJL model.

In order to clarify this point, let us turn to the analysis of the kinetic term (15) in the effective action of the NJL model in a magnetic field.

Because of the expression (22) for m_{dyn} at small g, the term $1/(\rho l)^2$ dominates in the functions $f_1^{00} = -f_1^{11}$ and $f_2^{00} = -f_2^{11}$, around the minimum $\rho = m_{dyn}$:

$$f_1^{00} = -f_1^{11} \simeq \frac{N_c}{8\pi^2} \frac{1}{\rho^2 l^2}, \quad f_2^{00} = -f_2^{11} \simeq -\frac{N_c}{24\pi^2} \frac{1}{\rho^2 l^2}. \tag{28}$$

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Up to the additional factor $|eB|/2\pi$, these functions coincide with those in (12) in the GN model. On the other hand, the functions $f_1^{22} = f_1^{33}$ and $f_2^{22} = f_2^{33}$, connected with derivatives with respect to the transverse coordinates, are strongly supressed, as compared to the functions (28), and the ratios of the functions $f_1^{22} = f_1^{33}$ and $f_2^{22} = f_2^{33}$ to those in (28) go rapidly (as m_{dyn}^2/Λ^2) to zero as $|eB| \to \infty$, $\Lambda^2/|eB| = C$.

As a result, the coordinates x_2 and x_3 become redundant variables in this limit: there are no transitions of field quanta between different points in the x_2x_3–plane. Therefore the model degenerates into a set of independent $(1+1)$–dimensional models, labeled by $\mathbf{x}_\perp = (x_2, x_3)$ coordinates. Let us show that all these models coincide with a $(1+1)$–dimensional GN model with the number of colors $\tilde{N}_c = (\pi/2C)N_c$, where the factor $C = \Lambda^2/|eB|$ was introduced above, in the definition of the "continuum" limit ($|eB| \to \infty$, $\Lambda^2/|eB| = C$).

Let us put the NJL model on a lattice with a the lattice spacing of the discretized space-time (in Euclidean region). Then its effective action can be written as

$$\Gamma_{NJL}(\sigma, \pi) = \int dx_2 dx_3 \int dx_4 dx_1 L_{NJL}^{(eff)}(\sigma(x), \pi(x))$$

$$\simeq \frac{1}{2\pi}|eB|a^4 \sum_{i,j=-\infty}^{\infty} \sum_{n,m=-\infty}^{\infty} \tilde{L}_{NJL}^{(eff)}(\sigma_{ij}(n,m), \pi_{ij}(n,m)) \qquad (29)$$

where $\sigma_{ij}(n,m) = \sigma(x)$, $\pi_{ij}(n,m) = \pi(x)$, with $x_2 = ia$, $x_3 = ja$, $x_4 = ix_0 = na$, $x_1 = ma$, and here the factor $|eB|/2\pi$ was explicitly factorized from $L_{NJL}^{(eff)}$. Now, taking into account Eqs.(7),(12) and Eqs.(28), (29), we find that

$$\Gamma_{NJL} = \frac{1}{2\pi}|eB|a^4 \sum_{i,j=-\infty}^{\infty} \sum_{n,m=-\infty}^{\infty} \tilde{L}_{NJL}^{(eff)}(\sigma_{ij}(n,m), \pi_{ij}(n,m))$$

$$\to \sum_{x_2,x_3} \int dx_4 dx_1 L_{GN}^{(eff)}(\sigma_{x_2x_3}(x_\parallel), \pi_{x_2x_3}(x_\parallel)) \qquad (30)$$

in the "continuum" limit with $(|eB|/2\pi)a^2 \equiv \pi|eB|/2\Lambda^2 = \pi/2C$ (here $\Lambda = \pi/a$ is the ultraviolet cutoff on the lattice)[3]. The lagrangian density $\mathcal{L}_{GN}^{(eff)}$ corresponds to the GN model with the number of colors $\tilde{N}_c = (\pi/2C)N_c$. Note that the symbol $\sum_{x_2,x_3}$ here is somewhat formal and it just implies that the GN model occurs at each point in the x_2x_3–plane.

The physical meaning of this reduction of the NJL model in a magnetic field is rather clear. At weak coupling, the fermion pairing in a magnetic field takes place essentially for fermions in the LLL with the momentum k_1=0. The size of the radius of the LLL orbit is $l=|eB|^{-1/2}$ [12]. As the magnetic field goes to infinity, this radius shrinks to zero. Then, because of the degeneracy in the LLL [12], there are $(|eB|/2\pi)a^2 = \pi/2C$ states with k_1=0 at each point in the x_2x_3–plane. This degeneracy factor (proportional to the (local) magnetic flux across a plaquette) leads to changing the number of colors, $N_c \to \tilde{N}_c = (\pi/2C)N_c$, in the GN model. Note that since $\tilde{N}_c$ appears analytically in the path integral of the theory, one can give a non-perturbative meaning to the theory with non-integral $\tilde{N}_c$.

A few comments are in order.

Since these GN models are independent, the parameters of the chiral $U_L(1) \times U_R(1)$ transformations can depend on $\mathbf{x}_\perp$. In other words, here the chiral group is $\prod_{x_\perp} U_L^{(x_\perp)}(1) \times U_R^{(x_\perp)}(1)$. As a result, there are an infinite number of gapless modes $\pi_{x_2x_3}(x_\parallel)$ in the "continuum" limit.

Since there is no spontaneous breakdown of continuous symmetries at $D = 1+1$, the fields $\pi_{x_2x_3}(x_\parallel)$ do not describe NG bosons (though they do describe gapless BKT excitations) [8].

Since the magnetic field depends on Λ in the "continuum" limit, it can be considered as an additional parameter ("coupling constant") in the renormalization group. The ratio $b = |eB|/\Lambda^2 = C^{-1}$ is arbitrary here. ¿From the point of view of the renormalization group, this can be interpreted as the presence of a line of ultraviolet fixed points for the dimensionless coupling b. The values of b on the line define the local magnetic flux and, therefore, the number of colors $\tilde{N}_c$ in the corresponding GN models.

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[3]Of course, the ultraviolet cutoff on the lattice is different from the cutoff in the proper-time regularization used above. However, since the constant $C = \Lambda^2/|eB|$ is anyway arbitrary here, we use the same notation for the cutoff on the lattice as in the proper-time regularization.

Our consideration of reducing the NJL model in a magnetic field to a continuum set of the GN models was somewhat heuristic. It would be worth deriving this reduction in a more rigorous way, putting the NJL model on a lattice in Euclidean space and then realizing renormalizations in the "continuum" limit.

In the next section, we shall discuss the connection between the GN model and the NJL model in a magnetic field in more detail.

IV. MORE ABOUT THE CONNECTION BETWEEN THE GN MODEL AND THE NJL MODEL IN A MAGNETIC FIELD

In the previous section, the reduction of the NJL model in a magnetic field of the infinite strength to a continuum set of the GN models was established. But what is the connection between the NJL and GN models at finite, though large, values of the magnetic field?

In order to answer this question, let us turn to a more detailed discussion of the infrared dynamics within these two models.

The GN model is asymptotically free, with the bare coupling constant $\tilde{G} = 2\pi/\tilde{N}_c \ln(\Lambda^2/m_{dyn}^2) \to 0$ as $\Lambda \to \infty$. Therefore there is dimensional transmutation in the model: in the scaling region ($\tilde{G} \ll 1$) the infrared dynamics, with momenta k satisfying $[\ln(\Lambda^2/k^2)]^{-1} \ll 1$, are essentially independent of either the coupling constant $\tilde{G}$ or the cutoff Λ; the only relevant parameter is the dynamical mass m_{dyn} (which is an analogue of the parameter Λ_{QCD} in QCD). Because of Eq.(22) for m_{dyn}, one might expect that a similar dimensional transmutation should take place in the NJL model in a magnetic field: for $|eB|GN_c \ll 1$, the infrared dynamics, with $k^2 \ll |eB|$, should be essentially independent of the magnetic field and the coupling G.

The real situation, however, is more subtle. Let us look at the propagators for fermions and σ and π particles at low momenta in the NJL model with a magnetic field, in leading order in $1/N_c$ [4]:

$$\tilde{S}^{(0)}(k) = i \exp\left(-\frac{\mathbf{k}_\perp^2}{|eB|}\right) \frac{k_0\gamma^0 - k_1\gamma^1 + m_{dyn}}{k_0^2 - k_1^2 - m_{dyn}^2} \left(1 - i\gamma^2\gamma^3\text{sign}(eB)\right), \tag{31}$$

$$\mathcal{D}_\sigma(k) = \frac{C_\sigma}{N_c} \left[k_0^2 - k_1^2 - \frac{3m_{dyn}^2}{|eB|} \ln\left(\frac{|eB|}{\pi m_{dyn}^2}\right) \mathbf{k}_\perp^2 - 12m_{dyn}^2\right]^{-1}, \tag{32}$$

$$\mathcal{D}_\pi(k) = \frac{C_\pi}{N_c} \left[k_0^2 - k_1^2 - \frac{m_{dyn}^2}{|eB|} \ln\left(\frac{|eB|}{\pi m_{dyn}^2}\right) \mathbf{k}_\perp^2\right]^{-1}, \tag{33}$$

where $\mathbf{k}_\perp = (k_2, k_3)$ and where C_σ and C_π are some inessential constants (we consider the case of the weak coupling, when relation (22) is valid). Because of relations (22) and (24), the coefficients of the $\mathbf{k}_\perp^2$–terms in the propagators $\mathcal{D}_\sigma(k)$ and $\mathcal{D}_\pi(k)$ are exponentially small. Also, in the infrared region, the fermion propagator is independent (up to power corrections $\sim (\mathbf{k}_\perp^2/|eB|)^n$, $n \geq 1$) of the magnetic field. Therefore one might think that in this case, like in the GN model, the only relevant parameter for the infrared dynamics is m_{dyn}. However, while the dependence of the propagators of fermions and σ particles on $|eB|$ can indeed be neglected, this dependence is essential in the case of the propagator of the gapless NG mode π. The $\mathbf{k}_\perp^2$–term in $\mathcal{D}_\pi(k)$ provides the $(3 + 1)$–dimensional character of the NG propagator which, as explained in Introduction, is crucial for the realization of spontaneous chiral symmetry breaking in the model. In a sense, the magnetic length $l = |eB|^{-1/2}$ plays here the same role as the ϵ–parameter in the $(2 + \epsilon)$–dimensional GN model [5].

As $|eB| \to \infty$, the NJL model is reduced to a set of independent GN models. However, at finite values of $|eB|$, the transverse velocity of the NG mode, though small $[|\mathbf{v}_\perp| \leq \sqrt{(m_{\text{dyn}}^2/|eB|)\ln(|eB|/\pi m_{\text{dyn}}^2)}]$, is not zero. Therefore there are now transitions of field quanta between different points in $x_2 x_3$–plane, i.e. interactions occur between the GN models associated with different points $\mathbf{x}_\perp$. As a result, at finite $|eB|$ the chiral group is reduced to global $U_L(1) \times U_R(1)$ transformations, and there is only one NG boson π for this case.

Therefore the reduction of the NJL model, described in the previous section, takes place only as $|eB| \to \infty$. At finite values of the magnetic field, the dynamics in the NJL and GN models are different: while there is spontaneous chiral symmetry breaking in the NJL model, the BKT phase is realized in the GN model [8]. The connection between these two sets of dynamics is similar to that between the dynamics of 2–dimensional and $(2 + \epsilon)$–dimensional GN models.

In conclusion, we emphasize that this discussion pertains only to the NJL model with a weak coupling constant, when relation (22) is valid. In the case of the NJL model with a near–critical g, the situation is different: when

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$g \to g_c - 0$, (23) is valid. The difference between these two dynamical regimes reflects the fact that, while at weak coupling the LLL dominates, at near–critical g all Landau levels are relevant [4].

In the next section, we shall discuss the dynamics in the supersymmetric NJL model.

V. CHIRAL SYMMETRY BREAKING IN THE SUPERSYMMETRIC NJL MODEL IN A MAGNETIC FIELD

As is well known, there is no spontaneous chiral symmetry breaking in the supersymmetric (SUSY) NJL model at any value of the coupling constant [6]. Here we shall show that an external magnetic field changes the situation dramatically: chiral symmetry breaking occurs for all (positive) values of the coupling constant. Moreover, at weak coupling, the dynamics in SUSY and ordinary NJL models are similar (and, therefore, intimately connected with the dynamics in the $(1+1)$–dimensional GN model).

The action of the SUSY NJL model with the $U_L(1) \times U_R(1)$ chiral symmetry in a magnetic field is

$$\Gamma = \int d^8z \left[\bar{Q}e^V Q + \bar{Q}^c e^{-V} Q^c + G(\bar{Q}^c\bar{Q})(QQ^c) \right]. \tag{34}$$

Here we utilize the notations of Ref. [13], except for our choice of metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In Eq.(34), $d^8z = d^4x d^2\theta d^2\bar{\theta}$, Q^α and Q^c_α are chiral superfields carrying the color index $\alpha = 1, 2, \ldots, N_c$, i.e. Q^α and Q^c_α are assigned to the fundamental and antifundamental representations of the $SU(N_c)$, respectively:

$$Q^\alpha = \varphi^\alpha + \sqrt{2}\theta\psi^\alpha + \theta^2 F^\alpha, \quad Q^c_\alpha = \varphi^c_\alpha + \sqrt{2}\theta\psi^c_\alpha + \theta^2 F^c_\alpha \tag{35}$$

(henceforth we will omit color indices). The vector superfield $V(x, \theta, \bar{\theta}) = -\theta\sigma^\mu\bar{\theta}A^{ext}_\mu$, with $A^{ext}_\mu = Bx^2\delta^3_\mu$, describes an external magnetic field which is in the $+x_1$ direction.

The action (34) is equivalent to the following action:

$$\Gamma_A = \int d^8z \left[\bar{Q}e^V Q + \bar{Q}^c e^{-V} Q^c + \frac{1}{G}\bar{H}H \right] -$$
$$- \int d^6z \left[\frac{1}{G}HS - QQ^cS \right] - \int d^6\bar{z} \left[\frac{1}{G}\bar{H}\bar{S} - \bar{Q}\bar{Q}^c\bar{S} \right]. \tag{36}$$

Here $d^6z = d^4x d^2\theta$, $d^6\bar{z} = d^4x d^2\bar{\theta}$, and H and S are two auxiliary chiral fields:

$$H = h + \sqrt{2}\theta\chi_h + \theta^2 f_h, \quad S = s + \sqrt{2}\theta\chi_s + \theta^2 f_s. \tag{37}$$

The Euler–Lagrange equations for these auxiliary fields take the form of constraints:

$$H = GQQ^c, \quad S = -\frac{1}{4}\bar{D}^2(\bar{H}) = -\frac{G}{4}\bar{D}^2(\bar{Q}\bar{Q}^c). \tag{38}$$

Here $\bar{D}$ is a SUSY covariant derivative [13]. The action (36) reproduces Eq.(34) upon application of the constraints (38).

In terms of the component fields, the action (36) is

$$\Gamma_A = \int d^4x \left[-\varphi^\dagger(\partial_\mu - ieA^{ext}_\mu)^2\varphi - \varphi^{c\dagger}(\partial_\mu + ieA^{ext}_\mu)^2\varphi^c \right.$$
$$+ i\bar{\psi}\bar{\sigma}^\mu(\partial_\mu - ieA^{ext}_\mu)\psi + i\bar{\psi}^c\bar{\sigma}^\mu(\partial_\mu + ieA^{ext}_\mu)\psi^c$$
$$+ F^\dagger F + F^{c\dagger}F^c + \frac{1}{G}\left(-h^\dagger\Box h + i\bar{\chi}_h\bar{\sigma}^\mu\partial_\mu\chi_h + f^\dagger_h f_h \right)$$
$$+ \frac{1}{G}\left(\chi_h\chi_s - hf_s - sf_h + \text{h.c.} \right)$$
$$\left. - \left(s\psi\psi^c + (\varphi\psi^c + \varphi^c\psi)\chi_s - s(\varphi F^c + \varphi^c F) - \varphi\varphi^c f_s + \text{h.c.} \right) \right]. \tag{39}$$

Let us consider the effective potential in this model. For this purpose, one can treat all the auxiliary scalar fields as (independent of x) constants and all the auxiliary fermion fields equal zero (since the auxiliary fields are colorless,

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loop diagrams involving them do not contribute to the effective potential in leading order in $1/N_c$). Then, using the Euler–Lagrange equations for the fields F, F^c, f_h, h and their conjugates, we find $F^\dagger = -s\varphi^c$, $F^{c\dagger} = -s\varphi$, $f_h^\dagger = s$, $f_s^\dagger = 0$, plus h.c. equations. Then the action becomes

$$\Gamma_A = \int d^4x \left[-\varphi^\dagger \left[(\partial_\mu - ieA_\mu^{ext})^2 + \rho^2 \right] \varphi - \varphi^{c\dagger} \left[(\partial_\mu + ieA_\mu^{ext})^2 + \rho^2 \right] \varphi^c \right.$$

$$\left. + i\bar\psi_D \gamma^\mu (\partial_\mu - ieA_\mu^{ext})\psi_D - \sigma\bar\psi_D\psi_D - \pi\bar\psi_D i\gamma^5\psi_D - \frac{\rho^2}{G} \right], \tag{40}$$

where $s = \sigma + i\pi$, $\rho^2 = |s|^2 = \sigma^2 + \pi^2$, and the Dirac fermion field ψ_D is introduced.

In leading order in $1/N_c$, the effective potential $V(\rho)$ can now be derived in the same way as in the ordinary NJL model [4]. The difference is that, besides fermions, the two scalar fields φ^c and φ give a contribution to $V(\rho)$:

$$V(\rho) = \frac{\rho^2}{G} + V_{fer}(\rho) + 2V_{bos}(\rho), \tag{41}$$

where

$$V_{fer}(\rho) = \frac{N_c}{8\pi^2 l^4} \int_{1/(l\Lambda)^2}^{\infty} \frac{ds}{s^2} \exp\left(-s(l\rho)^2\right) \coth s, \tag{42}$$

$$V_{bos}(\rho) = -\frac{N_c}{16\pi^2 l^4} \int_{1/(l\Lambda)^2}^{\infty} \frac{ds}{s^2} \exp\left(-s(l\rho)^2\right) \frac{1}{\sinh s}. \tag{43}$$

As is shown in the Appendix, the potential $V(\rho)$ can be rewritten as

$$V(\rho) = \frac{N_c}{8\pi^2 l^4} \left[\frac{(l\rho)^2}{g} + (l\rho)^2 \left(1 - \ln\frac{(l\rho)^2}{2} \right) + 4 \cdot \int_{(l\rho)^2/2}^{[(l\rho)^2+1]/2} dx \ln\Gamma(x) \right] +$$

$$+ \frac{N_c}{16\pi^2 l^4} \left[\ln(\Lambda l)^2 - \gamma - \ln(8\pi^2) \right] + O\left(\frac{1}{\Lambda}\right), \tag{44}$$

where the dimensionless coupling constant is $g = GN_c/8\pi^2 l^2$.

As the magnetic field B goes to zero ($l \to \infty$), we recover the known expression for the potential in the SUSY NJL model [6]:

$$V(\rho) = \frac{\rho^2}{G}. \tag{45}$$

The potential $V(\rho)$ is positive–definite, as has to be in a supersymmetric theory. The only minimum of this potential is $\rho = 0$ corresponding to the chiral symmetric vacuum.

The presence of a magnetic field changes this situation dramatically: at $B \neq 0$, a non–trivial global minimum, corresponding to spontaneous chiral symmetry breaking, exists for all $g > 0$. Moreover, we show below that, as the coupling $g \to 0$, the SUSY NJL model becomes equivalent to the ordinary NJL model and, therefore, at weak coupling and as $B \to \infty$, it is reduced to the same continuum set of the $(1+1)$–dimensional GN models. On the other hand, the dynamics of the SUSY and non–SUSY NJL models in a magnetic field are very different at strong coupling.

The gap equation $dV/d\rho = 0$, following from Eq.(44), is

$$\frac{N\rho}{4\pi^2 l^2} \left[\frac{1}{g} - \ln\frac{(l\rho)^2}{2} + 2\ln\Gamma\left(\frac{(l\rho)^2+1}{2} \right) - 2\ln\Gamma\left(\frac{(l\rho)^2}{2} \right) \right] = 0. \tag{46}$$

As can be seen from (44), at $B \neq 0$ the trivial solution $\rho = 0$ to this equation corresponds to a maximum of V: $d^2V/d\rho^2|_{\rho=0} = -\infty$. Numerical analysis of equation (46) for $g > 0$ and $B \neq 0$ shows that there is a nontrivial solution $\rho = \bar\sigma = m_{dyn}$ which is the global minimum of the potential. The analytic expression for m_{dyn} can be obtained at small g (when $m_{dyn}l \ll 1$) and very large g (when $m_{dyn}l \gg 1$). In those two cases, the results are:

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a) $g \ll 1$ ($m_{dyn}l \ll 1$). The gap equation (46) is

$$\frac{1}{g} \simeq - \ln \frac{\pi(\rho l)^2}{2},\tag{47}$$

i.e.

$$m_{dyn} \simeq \sqrt{\frac{2|eB|}{\pi}} \exp\left[-\frac{1}{2g}\right] = \sqrt{\frac{2|eB|}{\pi}} \exp\left[-\frac{4\pi^2}{|eB|N_cG}\right].\tag{48}$$

b) $g \gg 1$ ($m_{dyn}l \gg 1$). Now, the gap equation (46) gives

$$\frac{1}{g} \simeq \frac{1}{2(\rho l)^2},\tag{49}$$

i.e.

$$m_{dyn} \simeq \sqrt{\frac{g|eB|}{2}} = \frac{|eB|}{4\pi}\sqrt{GN_c}.\tag{50}$$

The numerical solution to the gap equation (46) for general values of g is shown in Fig. 1.

Let us discuss the case of the weak coupling in more detail. By using the asymptotic series for $\Gamma(x)$ [10], the potential (44) can be rewritten at $\rho l \ll 1$ as

$$V(\rho) = V(0) + \frac{N_c}{8\pi^2 l^4}\left[\frac{(\rho l)^2}{g} + (\rho l)^2\left(\ln\frac{(\rho l)^2}{2} - 1\right) + \right.$$

$$\left. + (\rho l)^2 \ln \pi + O((\rho l)^4)\right].\tag{51}$$

Using Eq.(47), the coupling constant g can be expressed through m_{dyn}, and the potential can be rewritten as

$$V(\rho) = V(0) + \frac{N_c\rho^2}{8\pi^2 l^2}\left[\ln\frac{\rho^2}{m_{dyn}^2} - 1 + O((\rho l)^2)\right].\tag{52}$$

This potential coincides (up to exponentially small terms) with that in the ordinary (weakly coupling) NJL model (see Eq.(27)).

Now, let us consider the kinetic term in the SUSY NJL model in a magnetic field. As shown in the Appendix, at weak coupling, this term also coincides with that in the ordinary NJL model. Therefore, at weak coupling, these two models are equivalent.

Note that the equivalence between the SUSY and non–SUSY NJL models becomes explicit only after the renormalization of the coupling constants. The structure of the ultraviolet divergences in the effective potentials of these models is quite different (compare Eqs. (14) and (44)). This reflects the point that only the infrared (and not ultraviolet) dynamics in these models are equivalent, i.e., in the renormalization group language, these two models are assigned to the same universality class [14].

The physical picture underlying this equivalence is clear. An external magnetic field explicitly breaks supersymmetry: the spectra of charged free fermions and bosons in a magnetic field are essentially different [12]. While for fermions the spectrum is

$$E_n(k_1) = \pm\sqrt{m^2 + 2|eB|n + k_1^2}, \qquad n = 0, 1, 2, \ldots,\tag{53}$$

for bosons it is

$$E_n(k_1) = \pm\sqrt{m^2 + |eB|(2n+1) + k_1^2}, \qquad n = 0, 1, 2, \ldots.\tag{54}$$

The crucial difference between them is that while for fermions the energy of the LLL is independent of B and (at $k_1 = 0$) goes to zero as $m \to 0$, for bosons the energy of the LLL is $E_0(k_1) = \pm\sqrt{|eB| + k_1^2}$. In other words, there is a gap $\Delta E = \sqrt{|eB|}$ in the spectrum of massless bosons in a magnetic field. Recall that, at weak coupling, the chiral

condensate in the NJL model occurs because of the fermion pairing in the LLL with $k_1 = 0$. This also happens in the SUSY NJL model, in which the bosonic degrees of freedom become irrelevant at weak coupling.

Notice that, in the effective action, the field ρ plays the role of the mass. In particular, the infrared singularities (as $\rho \to 0$) in the expressions for the potential and the kinetic term (see Eqs.(27) and (28)) reflect the absence of a gap in the LLL for massless fermions.

We emphasize that the equivalence between these two models takes place at weak coupling only. When g becomes larger than $g_c = 1$, the dynamics in these models are essentially different. While at $g > g_c = 1$ in the ordinary NJL model, the chiral symmetry is spontaneously broken even without an external magnetic field, there is no spontaneous chiral symmetry breaking in the SUSY NJL model as $B \to 0$ at any value of g (see Eq.(50)). The difference between the dynamical regimes with weak and strong coupling corresponds to the fact that, while at weak g the dynamics of the LLL of fermions dominates, at $g \geq g_c = 1$ all (fermionic and bosonic) Landau levels become relevant.

VI. CONCLUSION.

In this paper we have studied the infrared dynamics of both ordinary and SUSY NJL models in a magnetic field. It has been shown that, at weak coupling, the infrared dynamics in these two models are equivalent and, as $|eB| \to \infty$, the models reduce to a continuum set of $(1 + 1)$–dimensional GN models.

In this paper, as in Refs. [1–4], only the case of a homogeneous magnetic field has been considered. To extend the present results to inhomogeneous field configurations, we note first that the number of colors in the GN model is $\tilde{N}_c = (\pi/2C)N_c$, where the factor $\pi/2C = a^2|eB|/2\pi$ is proportional to the *local* magnetic flux attached to each point in $x_2 x_3$–plane. Let us consider a magnetic field $\mathbf{B}(\mathbf{x}_\perp)$, directed in $+x_1$ direction but depending on the transverse coordinates. It is tempting to speculate that in this case, as $|eB|(\mathbf{x}_\perp) \to \infty$, the NJL model will be reduced to a set of the GN models with the number of colors $\tilde{N}_c(\mathbf{x}_\perp) = (\pi/2C(\mathbf{x}_\perp))N_c$ (where $C(\mathbf{x}_\perp) = \Lambda^2/|eB|(\mathbf{x}_\perp)$), which is different in different points of the $x_2 x_3$-plane. It would be interesting to check these speculations by studying the NJL model in inhomogeneous field configurations.

The results of the present paper are in agreement with the general conclusion of Refs. [1–4], that the catalysis of chiral symmetry breaking by a magnetic field is a universal, model independent effect. This catalysis may have possible applications to cosmology, particle physics and condensed matter physics [4,15].

A specific application of interest would be a linkage between catalysis of chiral symmetry breaking and the existence of very strong primordial magnetic fields in the early universe [16]. The results of the present paper may be especially relevant for cosmological models based on supersymmetric dynamics [9].

Also, the effect of the dimensional reduction by external fields may be quite general and relevant for multi-dimensional field theories.

ACKNOWLEDGMENTS

APPENDIX: EFFECTIVE ACTION IN THE SUSY NJL MODEL

In this Appendix, the effective action for the SUSY NJL model in a magnetic field is derived.

Besides the expression for the fermion propagator in a magnetic field, which was used in Refs. [2,4], we also need to know the expression for the propagator in a magnetic field for a charged scalar with the mass $m = \rho$ [17]:

$$D(x, y) = \exp\left[\frac{ie}{2}(x - y)^\mu A_\mu^{\text{ext}}(x + y)\right]\tilde{D}(x - y) , \tag{A1}$$

where the Fourier transform of $\tilde{D}(x)$ is

$$\tilde{D}(k) = -\int_0^\infty \frac{ds}{\cosh eBs} \exp\left[-s\left(\rho^2 - k_0^2 + \mathbf{k}_\perp^2 \frac{\tanh(eBs)}{eBs} + k_3^2\right)\right]. \tag{A2}$$

The expression (41) for the effective potential in the SUSY NJL model can be obtained in the same way as for the ordinary NJL model in Ref. [4]. Let us show that this expression is equivalent to expression (44).

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Eq.(41) can be rewritten as

$$V(\rho) = \frac{\rho^2}{2G} + \frac{N_c}{16\pi^2 l^4}\left[\ln\left(\frac{\Lambda}{\rho}\right)^2 - \gamma + 2I((\rho l)^2)\right] + O\left(\frac{1}{\Lambda}\right) \tag{A3}$$

where

$$I(\beta) = \int_0^\infty ds\, e^{-\beta s}\left[\frac{\cosh s - 1}{s^2 \sinh s} - \frac{1}{2s}\right], \qquad \beta > 0. \tag{A4}$$

Let us show that

$$I(\beta) = \beta\left(1 - \ln\frac{\beta}{2}\right) + \frac{1}{2}\ln\beta - \frac{1}{2}\ln 8\pi^2 + 4\int_{\beta/2}^{(\beta+1)/2} dx\,\ln\Gamma(x). \tag{A5}$$

Using the integral representation for generalized zeta function [10], we find

$$I(\beta) = \lim_{\mu\to -1} I(\beta,\mu) \equiv \lim_{\mu\to -1}\int_0^\infty ds\, s^\mu e^{-\beta s}\left[\frac{\cosh s - 1}{s\sinh s} - \frac{1}{2}\right] =$$

$$= \lim_{\mu\to -1}\Gamma(\mu)\left[2^{1-\mu}\zeta\left(\mu,\frac{\beta}{2}\right) - \beta^{-\mu} - 2^{1-\mu}\zeta\left(\mu,\frac{\beta+1}{2}\right) - \frac{\mu}{2}\beta^{-1-\beta}\right]. \tag{A6}$$

In order to find this limit, we need the following identity:

$$\frac{\partial}{\partial z}\left[\zeta(z,q_2) - \zeta(z,q_1)\right]\Bigg|_{z=-1} = \frac{q_2 - q_1}{2}(q_2 + q_1 - 1 - \ln 2\pi) + \int_{q_1}^{q_2} dq\,\ln\Gamma(q). \tag{A7}$$

To derive it, we use the relation [10]

$$\frac{\partial}{\partial q}\zeta(z,q) = -z\zeta(z+1,q). \tag{A8}$$

Differentiating it with respect to z and integrating over q, we obtain

$$\frac{\partial}{\partial z}\left[\zeta(z,q_2) - \zeta(z,q_1)\right] = -\int_{q_1}^{q_2} dq\,\zeta(z+1,q) - z\int_{q_1}^{q_2} dq\,\frac{\partial}{\partial z}\zeta(z+1,q). \tag{A9}$$

Then, using the identities [10]

$$\zeta(0,q) = \frac{1}{2} - q, \qquad \frac{\partial}{\partial z}\zeta(z,q)\Bigg|_{z=0} = \ln\Gamma(q) - \frac{1}{2}\ln 2\pi, \tag{A10}$$

we obtain Eq.(A7).

¿From Eqs.(A6), (A7) and relations [10]

$$\Gamma(\mu) = \frac{\Gamma(\mu+2)}{(\mu+1)\mu}, \qquad \zeta(-1,q) = -\frac{1}{2}\left(q^2 - q + \frac{1}{6}\right), \tag{A11}$$

we obtain equation (A5).

Eqs.(A3) and (A5) lead to relation (44) for the effective potential.

Let us show that, at weak coupling, the kinetic term in the effective action of the SUSY NJL model coincides with that in the ordinary NJL model. For this purpose, we shall prove that the contribution of the scalar fields in the kinetic term is suppressed.

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By using the approach of Ref. [4], one finds that the functions $f_1^{\mu\nu}$ and $f_2^{\mu\nu}$ in the kinetic term (15) are now equal to:

$$f_1^{\mu\nu} = (f_1^{\mu\nu})_{fer}, \tag{A12}$$

$$f_2^{\mu\nu} = (f_2^{\mu\nu})_{fer} - 2i\rho^2 \int \frac{d^4k}{(2\pi)^4} \tilde{D}(k) \frac{\partial^2 \tilde{D}(k)}{\partial k_\mu \partial k_\nu}, \tag{A13}$$

where $(f_i^{\mu\nu})_{fer}$ are as given in (16). The crucial point for us is that, while the fermionic contribution (28) in $f_i^{\mu\nu}$ is divergent as $\rho l \to 0$, the contribution of the scalars is finite in this limit. This conclusion, following directly from Eqs.(A2) and (A13), reflects the fact that there is a gap in the spectrum of massless charged scalars in a magnetic field (see Eq.(54)). Since, at weak coupling, the parameter ρl is exponentially small, we conclude that the contribution of the scalars is indeed suppressed at small g.

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[16] M.S. Turner and L.M. Widrow, *Phys. Rev.* D**37,** 2743 (1988); T. Vachaspati, Phys. Lett. B**265,** 258 (1991); K. Enqvist and P. Olesen, Phys. Lett. B**319,** 178 (1993); J. M. Cornwall, in *Unified Symmetry: In the Small and Large*, eds. B. N. Kursunoglu *et al.* (Plenum Press, New York, 1995); G. Baym, D. Bodeker, and L. McLerran, *Phys. Rev.* D**53,** 662 (1996).
[17] J. Schwinger, Phys. Rev. **82,** 664 (1951).

FIGURE CAPTIONS

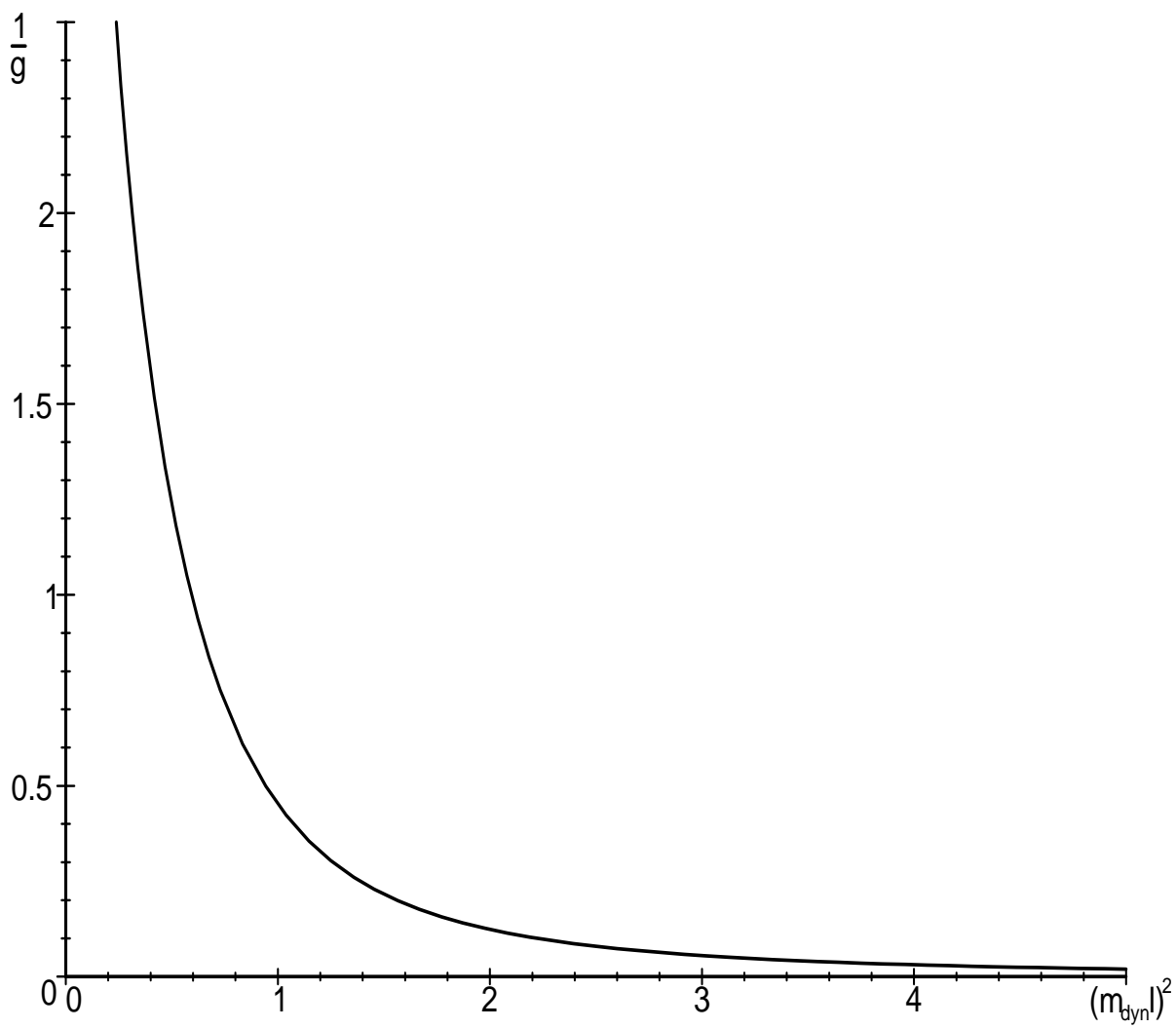Figure 1. The curve of $m_{dyn}l = m_{dyn}/\sqrt{|eB|}$ as the function of the inverse coupling constant $1/g$ in the SUSY NJL model.

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